LINEAR EQUATIONS WITH UNKNOWNS FROM A MULTIPLICATIVE GROUP WHOSE SOLUTIONS LIE IN A SMALL NUMBER OF SUBSPACES

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Abstract. Let K be a field of characteristic 0 and let $(K^*)^n$ denote the n-fold cartesian product of K^*, endowed with coordinatewise multiplication. Let Γ be a subgroup of $(K^*)^n$ of finite rank. We consider equations (*) $a_1x_1+\cdots+a_nx_n=1$ in $\mathbf{x}=(x_1,\dots,x_n)\in\Gamma$, where $\mathbf{a}=(a_1,\dots,a_n)\in(K^*)^n$. Two tuples $\mathbf{a},\mathbf{b}\in(K^*)^n$ are called Γ-equivalent if there is a $\mathbf{u}\in\Gamma$ such that $\mathbf{b}=\mathbf{u}\cdot\mathbf{a}$. Győry and the author [4] showed that for all but finitely many Γ-equivalence classes of tuples $\mathbf{a}\in(K^*)^n$, the set of solutions of (*) is contained in the union of not more than $2^{(n+1)!}$ proper linear subspaces of K^n. Later, this was improved by the author [3] to $(n!)^{2n+2}$. In the present paper we will show that for all but finitely many Γ-equivalence classes of tuples of coefficients, the set of non-degenerate solutions of (*) (i.e., with non-vanishing subsums) is contained in the union of not more than 2^n proper linear subspaces of K^n. Further we give an example showing that 2^n cannot be replaced by a quantity smaller than n.

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1. Introduction

Let K be a field of characteristic 0. Denote by $(K^*)^n$ the n-fold direct product of the multiplicative group K^*. The group operation of $(K^*)^n$ is coordinatewise multiplication, i.e., if $\mathbf{x}=(x_1,\dots,x_n)$, $\mathbf{y}=(y_1,\dots,y_n)\in(K^*)^n$, then $\mathbf{x}\cdot\mathbf{y}=(x_1y_1,\dots,x_ny_n)$. A subgroup Γ of $(K^*)^n$ is said to be of finite rank if there are $\mathbf{u}_1,\dots,\mathbf{u}_r\in\Gamma$ with the property that for every $\mathbf{x}\in\Gamma$ there are $z\in\mathbb{Z}_{>0}$ and $z_1,\dots,z_r\in\mathbb{Z}$ such that $\mathbf{x}^z=\mathbf{u}_1^{z_1}\cdots\mathbf{u}_r^{z_r}$. The smallest r for which such $\mathbf{u}_1,\dots,\mathbf{u}_r$

exist is called the rank of Γ; the rank of Γ is equal to 0 if all elements of Γ have finite order.

For the moment, let $n = 2$. We consider the equation

$$a_1x_1 + a_2x_2 = 1 \quad \text{in } \mathbf{x} = (x_1, x_2) \in \Gamma, \tag{1.1}$$

where $\mathbf{a} = (a_1, a_2) \in (K^*)^2$ and where Γ is a subgroup of $(K^*)^2$ of finite rank r. In 1996, Beukers and Schlickewei [2] showed that (1.1) has at most $2^{8(r+2)}$ solutions.

Two pairs $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ are called Γ-equivalent if there is an $\mathbf{u} \in \Gamma$ such that $\mathbf{b} = \mathbf{u} \cdot \mathbf{a}$. Clearly, two equations (1.1) with Γ-equivalent pairs of coefficients $\mathbf{a}$ have the same number of solutions. In 1988, Győry, Stewart, Tijdeman and the author [5] showed that there is a finite number of Γ-equivalence classes, such that for all tuples $\mathbf{a} = (a_1, a_2)$ outside the union of these classes, equation (1.1) has at most *two* solutions. (In fact they considered only groups $\Gamma = U_S \times U_S$ where U_S is the group of S-units in a number field, but their argument works in precisely the same way for the general case.) The upper bound 2 is best possible. We mention that this result is ineffective in that the method of proof does not allow to determine the exceptional equivalence classes. Bérczes [1, Lemma 3] calculated the upper bound $2e^{30^{20}(r+2)}$ for the number of exceptional equivalence classes.

Now let $n \geqslant 3$. We deal with equations

$$a_1x_1 + \cdots + a_nx_n = 1 \quad \text{in } \mathbf{x} = (x_1, \ldots, x_n) \in \Gamma, \tag{1.2}$$

where $\mathbf{a} = (a_1, \ldots, a_n) \in (K^*)^n$ and where Γ is a subgroup of $(K^*)^n$ of finite rank r. A solution $\mathbf{x}$ of (1.2) is called non-degenerate if

$$\sum_{i \in I} a_ix_i \neq 0 \quad \text{for each non-empty subset } I \text{ of } \{1, \ldots, r\}. \tag{1.3}$$

It is easy to show that there are groups Γ such that any degenerate solution of (1.2) gives rise to an infinite set of solutions. Schlickewei, Schmidt and the author [6] showed that equation (1.2) has at most $e^{(6n)^{3n}(r+1)}$ non-degenerate solutions. Their proof was based on a version of the quantitative Subspace Theorem, i.e., on the Thue-Siegel-Roth-Schmidt method. Recently, by a very different approach based on a method of Vojta and Faltings, Rémond [8] proved a general quantitative result for subvarieties of tori, which includes as a special case that for $n \geqslant 3$ equation (1.2) has at most $2^{n^{4n^2}(r+1)}$ non-degenerate solutions.

Two tuples $\mathbf{a}, \mathbf{b} \in (K^*)^n$ are called Γ-equivalent if $\mathbf{b} = \mathbf{u}\cdot\mathbf{a}$ for some $\mathbf{u} \in \Gamma$. Győry, Stewart, Tijdeman and the author [5] showed that for every sufficiently large r, there are a subgroup Γ of $(\mathbb{Q}^*)^n$ of rank r, and infinitely many Γ-equivalence classes of tuples $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{Q}^*)^n$, such that equation (1.2) has at least $e^{2r^{1/2}(\log r)^{-1/2}}$ non-degenerate solutions. This shows that in contrast to the case $n = 2$, for $n \geqslant 3$ there is no uniform bound C independent of Γ such that for all tuples $\mathbf{a}$ outside finitely many Γ-equivalence classes the number of non-degenerate solutions of (1.2) is at most C.

It turned out to be more natural to consider the minimal number m such that the set of solutions of (1.2) can be contained in the union of m proper linear subspaces of K^n. Notice that this minimal number m does not change if $\mathbf{a}$ is replaced by a Γ-equivalent tuple. In 1988 Győry and the author [4] showed that if K is a number field and $\Gamma = U_S^n$, i.e., the n-fold direct product of the group of S-units in K, then there are finitely many Γ-equivalence classes $C_1, \dots, C_t$ such that for every tuple $\mathbf{a} \in (K^*)^n \backslash (C_1 \cup \dots \cup C_t)$ the set of solutions of (1.2) is contained in the union of not more than $2^{(n+1)!}$ proper linear subspaces of K^n. This was improved by the author [3, Thm. 8] to $(n!)^{2n+2}$. Both the proofs of Győry and the author and that of the author can be extended easily to arbitrary fields K of characteristic 0 and arbitrary subgroups Γ of $(K^*)^n$ of finite rank.

For certain special groups Γ, Schlickewei and Viola [9, Corollary 2] improved the author's bound to $\binom{2n+1}{n} - n^2 - n - 2$. In fact, their result is valid for rank one groups $\Gamma = \{(\alpha_1^z, \dots, \alpha_n^z) : z \in \mathbb{Z}\}$, where $\alpha_1, \dots, \alpha_n$ are non-zero elements of a field K of characteristic 0 such that neither $\alpha_1, \dots, \alpha_n$, nor any of the quotients α_i/α_j $(0 \leqslant i < j \leqslant n)$ is a root of unity.

In the present paper we deduce a further improvement for the general equation (1.2).

Theorem. *Let K be a field of characteristic* 0*, let $n \geqslant 3$, and let Γ be a subgroup of $(K^*)^n$ of finite rank. Then there are finitely many Γ-equivalence classes $C_1, \dots, C_t$ of tuples in $(K^*)^n$, such that for every $\mathbf{a} = (a_1, \dots, a_n) \in (K^*)^n \backslash (C_1 \cup \dots \cup C_t)$, the set of non-degenerate solutions of*

$$a_1 x_1 + \dots + a_n x_n = 1 \quad \text{in } \mathbf{x} = (x_1, \dots, x_n) \in \Gamma \tag{1.2}$$

is contained in the union of not more than 2^n proper linear subspaces of K^n.

We mention that the set of degenerate solutions of (1.2) is contained in the union of at most $2^n - n - 2$ proper linear subspaces of K^n, each defined by a vanishing subsum $\sum_{i\in I} a_i x_i = 0$ where I is a subset of $\{1,\dots,n\}$ of cardinality $\neq 0,1,n$. So for $\mathbf{a} \notin C_1 \cup \cdots \cup C_t$, the set of (either degenerate or non-degenerate) solutions of (1.2) is contained in the union of at most $2^{n+1} - n - 2$ proper linear subspaces of K^n.

Our main tool is a qualitative finiteness result due to Laurent [7] for the number of non-degenerate solutions in Γ of a system of polynomial equations (or rather for the number of non-degenerate points in $X \cap \Gamma$ where X is an algebraic subvariety of the n-dimensional linear torus). Recently, Rémond [8] established for $K = \overline{\mathbb{Q}}$ an explicit upper bound for the number of these non-degenerate solutions. Using the latter, it is possible to compute a (very large) explicit upper bound for the number t of exceptional equivalence classes, depending on n and the rank r of Γ. We have not worked this out.

In Section 2 we recall Laurent's result. In Section 3 we prove our Theorem. In Section 4 we give an example showing that our bound 2^n cannot be improved to a quantity smaller than n.

2. Polynomial equations

Let as before K be a field of characteristic 0, let $n \geqslant 2$, and let $f_1,\dots,f_R \in K[X_1,\dots,X_n]$ be non-zero polynomials. Further, let Γ be a subgroup of $(K^*)^n$ of finite rank. We consider the system of equations

$$(2.1) \qquad f_i(x_1,\dots,x_n) = 0 \quad (i = 1,\dots,R) \quad \text{in } \mathbf{x} = (x_1,\dots,x_n) \in \Gamma.$$

Let λ be an auxiliary variable. A solution $\mathbf{x} = (x_1,\dots,x_n)$ of system (2.1) is called *degenerate* if there are integers $c_1,\dots,c_n$ with $\gcd(c_1,\dots,c_n) = 1$ such that

$$(2.2) \qquad f_i(\lambda^{c_1}x_1,\dots,\lambda^{c_n}x_n) = 0 \text{ identically in } \lambda \text{ for } i = 1,\dots,R$$

(meaning that by expanding the expressions, we get linear combinations of different powers of λ, all of whose coefficients are 0). Otherwise, the solution $\mathbf{x}$ is called *non-degenerate.*

Proposition 2.1. *System (2.1) has only finitely many non-degenerate solutions.*

Proof. Without loss of generality we may assume that K is algebraically closed. Let X denote the set of points $\mathbf{x} \in (K^*)^n$ with $f_i(\mathbf{x}) = 0$ for $i = 1, \dots, R$. By a result of Laurent [7, Théorème 2], the set of solutions $\mathbf{x} \in \Gamma$ of (2.1) is contained in the union of finitely many "families" $\mathbf{x}H = \{\mathbf{x} \cdot \mathbf{y} : \mathbf{y} \in H\}$, where $\mathbf{x} \in \Gamma$ and where H is an irreducible algebraic subgroup of $(K^*)^n$ such that $\mathbf{x}H \subset X$. [1]

Consider a family $\mathbf{x}H$ with $\mathbf{x} \in \Gamma$, $\mathbf{x}H \subset X$, $\dim H > 0$. Pick a one-dimensional irreducible algebraic group $H_0 \subset H$. There are integers $c_1, \dots, c_n$ with $\gcd(c_1, \dots, c_n) = 1$ such that $H_0 = \{(\lambda^{c_1}, \dots, \lambda^{c_n}) : \lambda \in K^*\}$. Then $\mathbf{x}H_0 = \{(x_0\lambda^{c_0}, \dots, x_n\lambda^{c_n}) : \lambda \in K^*\} \subset \mathbf{x}H \subset X$, and the latter implies (2.2). Conversely, if $\mathbf{x}$ satisfies (2.2) then $\mathbf{x}H_0 \subset X$. Therefore, the solutions of (2.1) contained in families $\mathbf{x}H$ with $\dim H > 0$ are precisely the degenerate solutions of (2.1). Each of the remaining families $\mathbf{x}H$, i.e., with $\dim H = 0$ consists of a single solution $\mathbf{x}$ since $H = \{(1, \dots, 1)\}$. It follows that system (2.1) has at most finitely many non-degenerate solutions. □

3. Proof of the Theorem

Let again K be a field of characteristic 0, let $n \geqslant 3$, and let Γ a subgroup of $(K^*)^n$ of finite rank. Further, let $\mathbf{a} = (a_1, \dots, a_n) \in (K^*)^n$. We deal with

$$a_1x_1 + \cdots + a_nx_n = 1 \quad \text{in } \mathbf{x} = (x_1, \dots, x_n) \in \Gamma. \tag{1.2}$$

Assume that (1.2) has a non-degenerate solution. By replacing $\mathbf{a}$ by a Γ-equivalent tuple we may assume that $\mathbf{1} = (1, \dots, 1)$ is a non-degenerate solution of (1.2). This means that

$$\begin{cases} a_1 + \cdots + a_n = 1, \\ \sum_{i \in I} a_i \neq 0 \text{ for each non-empty subset } I \text{ of } \{1, \dots, n\}. \end{cases} \tag{3.1}$$

We will show that there is a finite set of tuples $\mathbf{a}$ with (3.1) such that for each $\mathbf{a} \in (K^*)^n$ outside this set, the set of non-degenerate solutions of (1.2) is contained in the union of not more than 2^n proper linear subspaces of K^n. This clearly suffices to prove our Theorem.

[1]For $K = \overline{\mathbb{Q}}$, Rémond [8, Thm. 1] showed that the set of solutions of (2.1) is contained in the union of at most $(nd)^{n^3m^{3m^2}(r+1)}$ families $\mathbf{x}H$, where r is the rank of Γ, X has dimension m, and where each polynomial f_i has total degree $\leqslant d$. Probably his result can be extended to arbitrary fields K of characteristic 0 by means of a specialization argument.

By the result of Schlickewei, Schmidt and the author or that of Rémond mentioned in Section 1, there is a finite bound N independent of $\mathbf{a}$ such that equation (1.2) has at most N non-degenerate solutions. (In fact, already Győry and the author [4] proved the existence of such a bound but their method did not allow to compute it explicitly).

For every tuple $\mathbf{a}$ with (3.1), we make a sequence $\mathbf{x}_1 = \mathbf{1}$, $\mathbf{x}_2 = (x_{21}, \dots, x_{2n}), \dots,$ $\mathbf{x}_N = (x_{N1}, \dots, x_{Nn})$ such that each term $\mathbf{x}_i$ is a non-degenerate solution of (1.2) and such that each non-degenerate solution of (1.2) occurs at least once in the sequence. Then

$$\mathrm{rank}\begin{pmatrix} 1 & \cdots & 1 & 1 \\ x_{21} & \cdots & x_{2n} & 1 \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ x_{N,1} & \cdots & x_{N,n} & 1 \end{pmatrix} \leqslant n \tag{3.2}$$

since the matrix has $n+1$ linearly dependent columns. Relation (3.2) means that the determinants of all $(n+1)\times(n+1)$-submatrices of the matrix on the left-hand side are 0. Thus, we may view (3.2) as a system of polynomial equations of the shape (2.1), to be solved in $(\mathbf{x}_2, \dots, \mathbf{x}_N) \in \Gamma^{N-1}$. It is important to notice that this system is independent of $\mathbf{a}$.

The tuples $\mathbf{a}$ with (3.1) are now divided into three classes:

Class I consists of those tuples $\mathbf{a}$ such that $\mathrm{rank}\,\{\mathbf{1}, \mathbf{x}_2, \dots, , \mathbf{x}_N\} = n$ and such that $(\mathbf{x}_2, \dots, \mathbf{x}_N)$ is a non-degenerate solution in Γ^{N-1} of system (3.2).

Class II consists of those tuples $\mathbf{a}$ such that $\mathrm{rank}\,\{\mathbf{1}, \mathbf{x}_2, \dots, , \mathbf{x}_N\} < n$.

Class III consists of those tuples $\mathbf{a}$ such that $(\mathbf{x}_2, \dots, \mathbf{x}_N)$ is a degenerate solution in Γ^{N-1} of system (3.2).

First let $\mathbf{a}$ be a tuple of Class I. By Proposition 2.1, $(\mathbf{x}_2, \dots, \mathbf{x}_N)$ belongs to a finite set which is independent of $\mathbf{a}$. Now $\mathbf{a} = (a_1, \dots, a_n)$ is a solution of the system of linear equations $a_1 + \cdots + a_n = 1$, $x_{i1}a_1 + \cdots + x_{in}a_n = 1$ $(i = 2, \dots, N)$. Since by assumption, $\mathrm{rank}\,\{\mathbf{1}, \mathbf{x}_2, \dots, \mathbf{x}_N\} = n$, the tuple $\mathbf{a}$ is uniquely determined by $\mathbf{x}_2, \dots, \mathbf{x}_N$. So Class I is finite.

For tuples $\mathbf{a}$ from Class II, all non-degenerate solutions of (1.2) lie in a single proper subspace of K^n.

Now let $\mathbf{a}$ be from Class III. In view of (2.2) this means that there are integers c_{ij} ($i = 2,\dots,N$, $j = 1,\dots,n$), with $\gcd(c_{ij} : i = 2,\dots,N,\ j = 1,\dots,n) = 1$, such that

$$\operatorname{rank}\begin{pmatrix} 1 & \cdots & 1 & 1 \\ \lambda^{c_{21}}x_{21} & \cdots & \lambda^{c_{2n}}x_{2n} & 1 \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ \lambda^{c_{N,1}}x_{N,1} & \cdots & \lambda^{c_{N,n}}x_{N,n} & 1 \end{pmatrix} \leqslant n$$

identically in λ, meaning that the determinants of the $(n+1)\times(n+1)$-submatrices of the left-hand side are identically zero in λ.

This implies that there are rational functions $b_j(\lambda) \in K(\lambda)$ ($j = 0,\dots,n$), not all equal to 0, such that

$$\sum_{j=1}^{n} b_j(\lambda) = b_0(\lambda), \quad \sum_{j=1}^{n} b_j(\lambda)\lambda^{c_{ij}}x_{ij} = b_0(\lambda) \ \ (i = 2,\dots,N)\,. \tag{3.3}$$

By clearing denominators, we may assume that $b_0(\lambda),\dots,b_n(\lambda)$ are polynomials in $K[\lambda]$ without a common zero.

We substitute $\lambda = -1$. Put $b_j := b_j(-1)$ ($j = 0,\dots,n$) and $\varepsilon_{ij} := (-1)^{c_{ij}}$ ($i = 2,\dots,N$, $j = 1,\dots,n$). Then $(b_0,\dots,b_n) \neq (0,\dots,0)$, and the numbers ε_{ij} are not all equal to 1 since the integers c_{ij} are not all even. Further, by (3.3) we have

$$\begin{cases} b_1 + \cdots + b_n = b_0\,, \\ b_1\varepsilon_{i1}x_{i1} + \cdots + b_n\varepsilon_{in}x_{in} = b_0 \quad \text{for } i = 2,\dots,N. \end{cases} \tag{3.4}$$

We claim that for each tuple $(\varepsilon_1,\dots,\varepsilon_n) \in \{-1,1\}^n$, the tuple $(b_1\varepsilon_1,\dots,b_n\varepsilon_n,b_0)$ is not proportional to $(a_1,\dots,a_n,1)$. Assuming this to be true, it follows from (3.4) that the set of non-degenerate solutions of (1.2) is contained in the union of at most 2^n proper linear subspaces of K^n, each given by

$$b_0\Big(\sum_{j=1}^{n} a_jx_j\Big) - \sum_{j=1}^{n} b_j\varepsilon_jx_j = 0$$

for certain $\varepsilon_j \in \{-1,1\}$ ($j = 1,\dots,n$).

We prove our claim. First suppose that the tuple $(b_1,\dots,b_n,b_0)$ is proportional to $(a_1,\dots,a_n,1)$. There are $i\in\{2,\dots,N\}$, $j\in\{1,\dots,n\}$ such that $\varepsilon_{ij}=-1$. Now $\mathbf{x}_i$ satisfies both $\sum_{j=1}^n a_jx_{ij}=1$ (since it is a solution of (1.2)) and $\sum_{j=1}^n a_j\varepsilon_{ij}x_{ij}=1$ (by (3.4)). But then by subtracting we obtain $\sum_{j\in J}a_jx_{ij}=0$, where J is the set of indices j with $\varepsilon_{ij}=-1$. This is impossible since $\mathbf{x}_i$ is a non-degenerate solution of (1.2).

Now suppose that $(b_1\varepsilon_1,\dots,b_n\varepsilon_n,b_0)$ is proportional to $(a_1,\dots,a_n,1)$ for certain $\varepsilon_j\in\{-1,1\}$, not all equal to 1. Then by (3.1) and (3.4) we have $\sum_{j=1}^n a_j=1$, $\sum_{j=1}^n a_j\varepsilon_j=1$. Again by subtracting, we obtain $\sum_{j\in J}a_j=0$ where J is the set of indices j with $\varepsilon_j=-1$ and this is contradictory to (3.1). This proves our claim.

Summarizing, we have proved that Class I is finite, that for every $\mathbf{a}$ in Class II, all solutions of (1.2) lie in a single proper linear subspace of K^n, and that for every $\mathbf{a}$ in Class III, the solutions of (1.2) lie in the union of 2^n proper linear subspaces of K^n. Our Theorem follows. □

4. Equations whose solutions lie in many subspaces

We give an example of a group Γ with the property that there are infinitely many Γ-equivalence classes of tuples $\mathbf{a}=(a_1,\dots,a_n)\in(K^*)^n$ such that the set of non-degenerate solutions of (1.2) cannot be covered by fewer than n proper linear subspaces of K^n.

Let K be a field of characteristic 0, let $n\geqslant 2$, and let Γ_1 be an infinite subgroup of K^* of finite rank. Take $\Gamma:=\Gamma_1^n=\{\mathbf{x}=(x_1,\dots,x_n):x_i\in\Gamma_1\text{ for }i=1,\dots,n\}$. Then Γ is a subgroup of $(K^*)^n$ of finite rank.

Pick $\mathbf{u}=(u_1,\dots,u_n)\in\Gamma$ with $b:=u_1+\cdots+u_n\neq 0$ and with $\sum_{i\in I}u_i\neq 0$ for each non-empty subset I of $\{1,\dots,n\}$. Let S_n denote the group of permutations of $\{1,\dots,n\}$. For $\sigma\in S_n$ write $\mathbf{u}_\sigma:=(u_{\sigma(1)},\dots,u_{\sigma(n)})$. Then $\mathbf{u}_\sigma$ ($\sigma\in S_n$) are non-degenerate solutions of

$$b^{-1}x_1+\cdots+b^{-1}x_n=1\quad\text{in }\mathbf{x}\in\Gamma. \tag{4.1}$$

For $i=1,\dots,n$, the points $\mathbf{u}_\sigma$ with $\sigma(n)=i$ lie in the subspace given by

$$u_i(x_1+\cdots+x_{n-1})-(b-u_i)x_n=0.$$

Therefore, for fixed $\mathbf{u}$, the set $\{\mathbf{u}_\sigma : \sigma \in S_n\}$ can be covered by n subspaces. We show that for "sufficiently general" $\mathbf{u}$, this set cannot be covered by fewer than n subspaces.

We need some auxiliary results.

Lemma 4.1. *Let $n \geqslant 2$ and let S be a subset of S_n of cardinality $> (n-1)!$. Then there are $\sigma_1, \ldots, \sigma_n \in S$ such that the polynomial*

$$F_{\sigma_1,\ldots,\sigma_n}(X_1,\ldots,X_n) := \begin{vmatrix} X_{\sigma_1(1)} & \cdots & X_{\sigma_1(n)} \\ X_{\sigma_2(1)} & \cdots & X_{\sigma_2(n)} \\ \vdots & & \vdots \\ X_{\sigma_n(1)} & \cdots & X_{\sigma_n(n)} \end{vmatrix} \tag{4.2}$$

is not identically zero.

Proof. We proceed by induction on n. For $n = 2$ the lemma is trivial. Assume that $n \geqslant 3$.

First assume there are $i, j \in \{1, \ldots, n\}$ such that the set $S_{ij} = \{\sigma \in S : \sigma(i) = j\}$ has cardinality $> (n-2)!$. Then after a suitable permutation of the columns of the determinant of (4.2) and a permutation of the variables $X_1, \ldots, X_n$, we obtain that S_{nn} has cardinality $> (n-2)!$. The elements of S_{nn} permute $1, \ldots, n-1$. Therefore, by the induction hypothesis, there are $\sigma_1, \ldots, \sigma_{n-1} \in S_{nn}$ such that the polynomial

$$G(X_1,\ldots,X_{n-1}) := \begin{vmatrix} X_{\sigma_1(1)} & \cdots & X_{\sigma_1(n-1)} \\ \vdots & & \vdots \\ X_{\sigma_{n-1}(1)} & \cdots & X_{\sigma_{n-1}(n-1)} \end{vmatrix}$$

is not identically zero. Since S_{nn} has cardinality $\leqslant (n-1)!$, there is a $\sigma_n \in S$ with $\sigma_n(n) = k \neq n$. Therefore,

$$F_{\sigma_1,\ldots,\sigma_n}(X_1,\ldots,X_{n-1},0) = \pm X_k \cdot G(X_1,\ldots,X_{n-1}) \neq 0.$$

So in particular, $F_{\sigma_1,\ldots,\sigma_n}$ is not identically zero.

Now suppose that for each pair $i, j \in \{1, \ldots, n\}$ the set S_{ij} has cardinality $\leqslant (n-2)!$. Together with our assumption that S has cardinality $> (n-1)!$, this implies that $S_{ij} \neq \emptyset$ for $i, j \in \{1, \ldots, n\}$. Thus, we may pick $\sigma_1 \in S$ with $\sigma_1(1) = 1$, $\sigma_2 \in S$ with $\sigma_2(2) = 1, \ldots, \sigma_n \in S$ with $\sigma_n(n) = 1$. Then $F_{\sigma_1,\ldots,\sigma_n}(1, 0, \ldots, 0) = 1$, hence $F_{\sigma_1,\ldots,\sigma_n}$ is not identically zero. □

Let T denote the collection of tuples $(\sigma_1, \dots, \sigma_n)$ in S_n for which $F_{\sigma_1,\dots,\sigma_n}$ is not identically 0. Let B be the set of numbers of the shape $u_1 + \cdots + u_n$ where $\mathbf{u} = (u_1, \dots, u_n)$ runs through all tuples in $\Gamma = \Gamma_1^n$ with

$$\left\{ \begin{array}{l} \sum_{i \in I} u_i \neq 0 \quad \text{for each } I \subseteq \{1, \dots, n\} \text{ with } I \neq \emptyset; \\ F_{\sigma_1,\dots,\sigma_n}(u_1, \dots, u_n) \neq 0 \quad \text{for each } (\sigma_1, \dots, \sigma_n) \in T. \end{array} \right. \tag{4.3}$$

In particular (taking $I = \{1, \dots, n\}$), each $b \in B$ is non-zero.

Two numbers $b_1, b_2 \in K^*$ are called Γ_1-equivalent if $b_1/b_2 \in \Gamma_1$.

Lemma 4.2. *The set B is not contained in the union of finitely many Γ_1-equivalence classes.*

Proof. First suppose that $B \neq \emptyset$. Assume that B is contained in the union of finitely many Γ_1-equivalence classes. Let $b_1, \dots, b_t$ be representatives for these classes. Then for every $\mathbf{u} = (u_1, \dots, u_n) \in \Gamma$ with (4.3) there are $b_i \in \{b_1, \dots, b_t\}$ and $u \in \Gamma_1$ such that

$$u_1 + \cdots + u_n = b_i u.$$

Hence for given b_i, $(u_1/u, \dots, u_n/u)$ is a non-degenerate solution of

$$x_1 + \cdots + x_n = b_i \quad \text{in } \mathbf{x} = (x_1, \dots, x_n) \in \Gamma.$$

Each such equation has only finitely many non-degenerate solutions. Therefore, for each b_i there are only finitely many possibilities for $(u_1/u, \dots, u_n/u)$, hence only finitely many possibilities for u_1/u_2. So if $(u_1, \dots, u_n)$ runs through all tuples in Γ with (4.3), then u_1/u_2 runs through a finite set, U, say.

Now let F be the product of the polynomials $F_{\sigma_1,\dots,\sigma_n}$ $((\sigma_1, \dots, \sigma_n) \in T)$, $\sum_{i \in I} X_i$ $(I \subseteq \{1, \dots, n\}, I \neq \emptyset)$ and $X_1 - uX_2$ $(u \in U)$. Then $F(u_1, \dots, u_n) = 0$ for every $u_1, \dots, u_n \in \Gamma_1$. But since Γ_1 is infinite, this implies that F is identically zero. Thus, if we assume that $B \neq \emptyset$ and that Lemma 4.2 is false we obtain a contradiction. The assumption $B = \emptyset$ leads to a contradiction in a similar manner, taking for F the product of the polynomials $F_{\sigma_1,\dots,\sigma_n}$ $((\sigma_1, \dots, \sigma_n) \in T)$, $\sum_{i \in I} X_i$ $(I \subseteq \{1, \dots, n\}, I \neq \emptyset)$. $\square$

Lemma 4.2 implies that the collection of tuples $(b^{-1},\dots,b^{-1})$ (n times) with $b\in B$ is not contained in the union of finitely many Γ-equivalence classes. We show that for every $b\in B$, the set of non-degenerate solutions of (4.1) cannot be covered by fewer than n proper linear subspaces of K^n.

Choose $b\in B$, and choose $\mathbf{u}=(u_1,\dots,u_n)\in\Gamma$ with $u_1+\cdots+u_n=b$ and with (4.3). Then each vector $\mathbf{u}_\sigma$ ($\sigma\in S_n$) is a non-degenerate solution of (4.1).

We claim that a proper linear subspace of K^n cannot contain more than $(n-1)!$ vectors $\mathbf{u}_\sigma$ ($\sigma\in S_n$). For suppose some subspace L of K^n contains more than $(n-1)!$ vectors $\mathbf{u}_\sigma$. Then by Lemma 4.1, there are $\sigma_1,\dots,\sigma_n\in S_n$ such that $\mathbf{u}_{\sigma_i}\in L$ for $i=1,\dots,n$ and such that $F_{\sigma_1,\dots,\sigma_n}$ is not identically 0. But since $\mathbf{u}$ satisfies (4.3), we have $F_{\sigma_1,\dots,\sigma_n}(\mathbf{u})\neq 0$. Therefore, the vectors $\mathbf{u}_{\sigma_1},\dots,\mathbf{u}_{\sigma_n}$ are linearly independent. Hence $L=K^n$.

Our claim shows that at least n proper linear subspaces of K^n are needed to cover the set $\mathbf{u}_\sigma$ ($\sigma\in S_n$). Therefore, the set of non-degenerate solutions of (4.1) cannot lie in the union of fewer than n proper subspaces.

References

[1] A. Bérczes, *On the number of solutions of norm form equations*, Period. Math. Hungar. 43 (2001), 165-176.

[2] F. Beukers, H.P. Schlickewei, *The equation $x+y=1$ in finitely generated groups*, Acta Arith. 78 (1996), 189-199.

[3] J.-H. Evertse, *Decomposable form equations with a small linear scattering*, J. reine angew. Math. 432 (1992), 177–217.

[4] J.-H. Evertse, K. Győry, *On the numbers of solutions of weighted unit equations*, Compos. Math. 66 (1988), 329–354.

[5] J.-H. Evertse, K. Győry, C.L. Stewart, R. Tijdeman, *S-unit equations in two unknowns*, Invent. Math. 92 (1988), 461–477.

[6] J.-H. Evertse, H.P. Schlickewei, W.M. Schmidt, *Linear equations in variables which lie in a multiplicative group*, Ann. Math. 155 (2002), 807-836.

[7] M. Laurent, *Équations diophantiennes exponentielles*, Invent. Math. 78 (1984), 299–327.

[8] G. Rémond, *Sur les sous-variétés des tores*, Compos. Math. 134 (2002), 337-366.

[9] H.P. Schlickewei, C. Viola, *Generalized Vandermonde determinants*, Acta Arith. 95 (2000), 123-137.

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